Solutions

AQA AS-Level Further Maths 2026 Paper 1

Do nut turn over the page until instructed to do so.

This assessment is out of 80 marks and you will be given 90 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: / 80

1 Which of the following is a vector which is not perpendicular to the line with vector equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda\left(2\mathbf{i} + 4\mathbf{j} - \mathbf{k}\right)$?

$\begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 0 \\ -1 \\ -4 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

[1 mark]

$\begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 0$ $\quad$ $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 12$

$\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 0$

$\begin{pmatrix} 0 \\ -1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 0$

2 $\mathbf{A}$ is a square $n \times n$ matrix.
If $\det(\mathbf{A}) = 6$ what is $\det\left(\mathbf{A}^{-1}\right)$?

6 $\quad$ $\frac{1}{6}$ $\quad$ 36 $\quad$ 1

[1 mark]

3 Let $z = 3 + 2\mathrm{i}$ be a root of $p(z) = z^2 - (5 + \mathrm{i})z + (8 + i)$. Which is the other root?

$3 - 2\mathrm{i}$ $\quad$ $-3 - 2\mathrm{i}$ $\quad$ $2 - \mathrm{i}$ $\quad$ $2 + \mathrm{i}$

[1 mark]

$\sum \alpha = 5 + i$

So $\quad 3 + 2i + (a + bi) = 5 + i$

$\underline{Re}$ $\quad 3 + a = 5 \quad \Rightarrow a = 2$

$\underline{Im}$ $\quad 2 + b = 1 \quad \Rightarrow b = -1$

4 Find $\sum_{r=3}^{10} r^2$

380 $\quad$ 371 $\quad$ 385 $\quad$ 384

[1 marks]

5 a) Using the standard Maculaurin expansions find the first three terms in the series expansion of $y = \cosh(ix)$.

[4 marks]

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!}$$

$$= 1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \frac{x^4}{24}$$

So

$$e^{-ix} = 1 - ix - \frac{x^2}{2} + \frac{ix^3}{6} + \frac{x^4}{24}$$

So

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2}$$

$$= \frac{2 - x^2 + \frac{2x^4}{24}}{2}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{2\cancel{4}} - \ldots$$

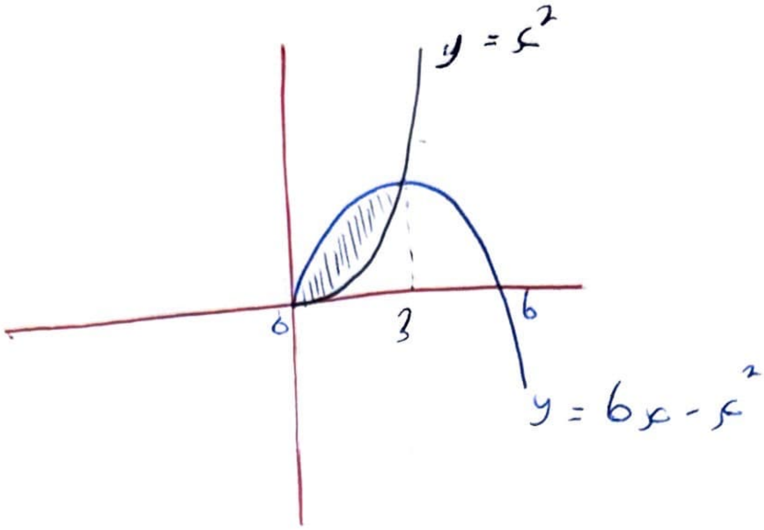
b) Hence, suggest an identity relating $\cosh(ix)$ to a trigonometric function.

[1 mark]

$$\cosh(ix) = \cos(x)$$

6 Find the volume of the solid formed when the region enclosed between $y = x^2$ and $y = 6x - x^2$ is rotated through $360°$ about the x-axis.

[5 marks]



So using

$$V = \pi \int y^2 \, dx$$

watch out for this - this is a very common mistake

$$= \cancel{\pi \int \left((6x - x^2) - x^2 \right)^2 dx}$$

$$= \pi \int_0^3 (6x - x)^2 - (x^2)^2 \, dx$$

$$= \pi \int_0^3 36x^2 - 12x^3 + x^4 - x^4 \, dx$$

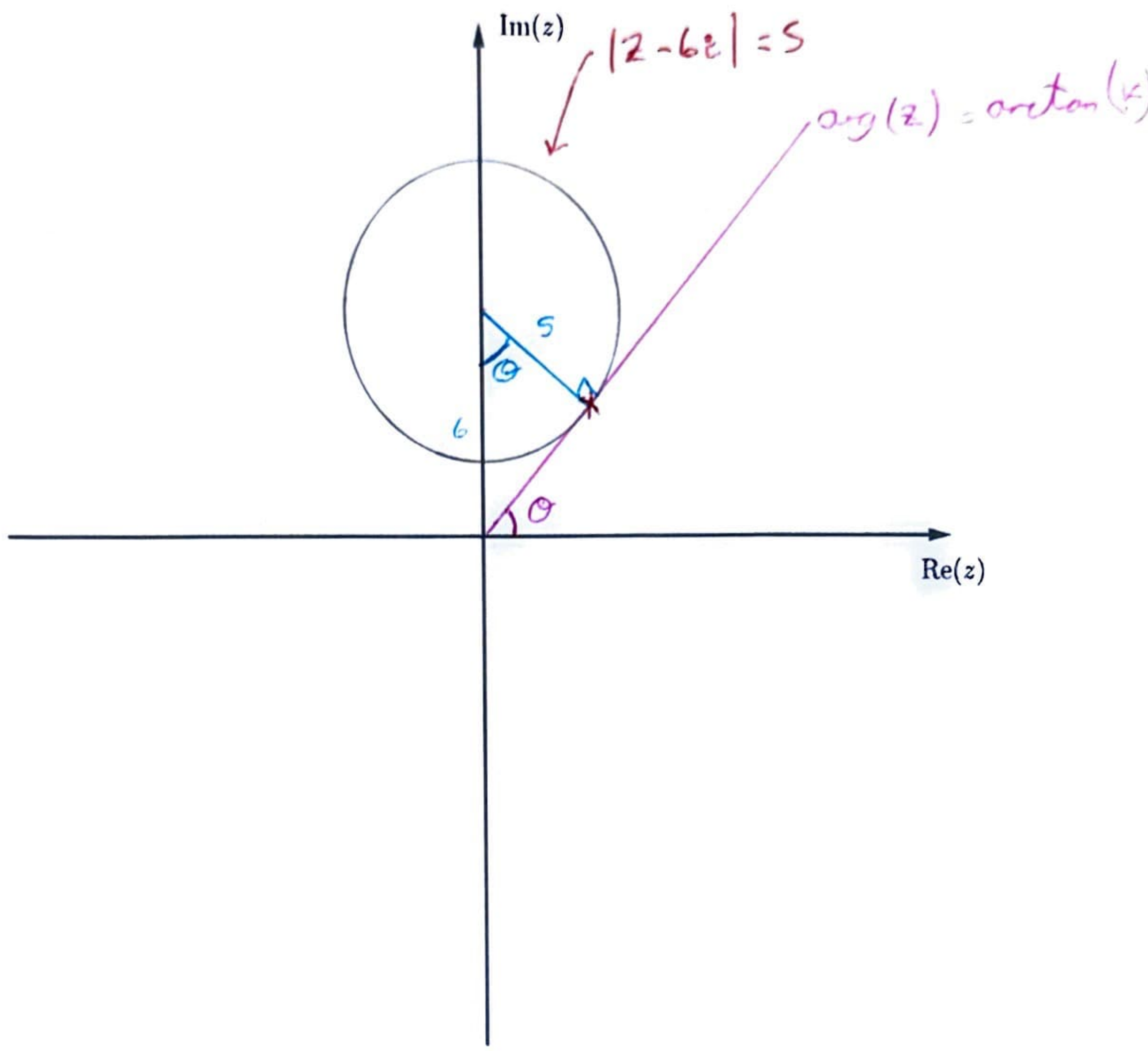
$$= \pi \int_0^3 36x^2 - 12x^3 \, dx$$

$$= \pi \left[12x^3 - 3x^4 \right]_0^3$$

$= 81\pi$ cubic units

7 a) Sketch, on the argand diagram below, the locus of the points which satisfy $|z - 6i| = 5$

[2 marks]



b) Point P is the only point that lies on both C and the half line $\arg(z) = \arctan(k)$, where k is a positive constant.

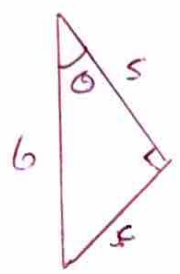
Find the value of k, giving your answer in exact form.

[3 marks]

With reference to the diagram

$x = \sqrt{6^2 - 5^2}$

$= \sqrt{11}$



$\theta = \arctan\left(\frac{\sqrt{11}}{5}\right)$

So $k = \dfrac{\sqrt{11}}{5}$

c) Find, in the form $a + bi$ the coordinates of the point P, giving the values of a and b to 3 significant figures.

[3 marks]

$x = \sqrt{11}\cos\left(\arctan\left(\frac{\sqrt{11}}{5}\right)\right)$

$= \dfrac{5\sqrt{11}}{6}$

$y = \sqrt{11}\sin\left(\arctan\left(\frac{\sqrt{11}}{5}\right)\right)$

$= \dfrac{11}{6}$

Hence $P\left(\frac{5\sqrt{11}}{6}, \frac{11}{6}\right)$

8) a) Prove that

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

[4 marks]

Let $y = \operatorname{arsinh}(x)$, then

$\sinh(y) = x$

$\Rightarrow \dfrac{e^y - e^{-y}}{2} = x$

$\Rightarrow e^y - 2x - e^{-y} = 0$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0$

$\Rightarrow (e^y - x)^2 - x^2 - 1 = 0$

$\Rightarrow (e^y - x)^2 = 1 + x^2$

$e^y - x = \pm\sqrt{1+x^2}$

$e^y = x \pm \sqrt{x^2+1}$

So

$e^y = x + \sqrt{x^2+1}$ since otherwise LHS > 0 and RHS < 0.

so $y = \ln\left(x + \sqrt{x^2+1}\right)$

i.e, $\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

b) Find, in exact form, the solutions of the equation,

$$2\cosh^2(x) + \sinh(x) = 5$$

[4 marks]

$\cosh^2(x) - \sinh^2(x) = 1$

$\Rightarrow \cosh^2(x) = 1 + \sinh^2(x)$

So,

$2\cosh^2(x) + \sinh(x) = 5$

$\Rightarrow 2(1 + \sinh^2(x)) + \sinh(x) - 5 = 0$

$\Rightarrow 2\sinh^2(x) + \sinh(x) - 3 = 0$

$(2\sinh(x) + 3)(\sinh(x) - 1)$

so

$\sinh(x) = -\dfrac{3}{2}$ or $\sinh(x) = 1$

Using (a)

$x = \ln\left(-\dfrac{3}{2} + \sqrt{\left(-\dfrac{3}{2}\right)^2 + 1}\right)$ or $x = \ln\left(1 + \sqrt{1^2 + 1}\right)$

$= \ln\left(1 + \sqrt{2}\right)$

$= \ln\left(-\dfrac{3}{2} + \sqrt{\dfrac{13}{4}}\right)$

$= \ln\left(\dfrac{-3}{2} + \dfrac{\sqrt{13}}{2}\right)$

$= \ln\left(\sqrt{13} - 3\right)$

9 a) Show that $(r+2)^2 - (r-1)^2 = 6r + 3$

[1 marks]

$$LHS = (r+2)^2 - (r-1)^2$$
$$= r^2 + 4r + 4 - (r^2 - 2r + 1)$$
$$= 6r + 3$$

b) Hence, using the method of differences, find an expression for

$$\sum_{r=1}^{n} 2r + 1$$

[4 marks]

$$\sum_{r=1}^{n} 2r+1 = \frac{1}{3}\sum (r+2)^2 - (r-1)^2$$

$= \frac{1}{3}[\cancel{+3^2} - 0^2$	$r=1$
$+\cancel{4^2} - 1^2$	$r=2$
$+\cancel{5^2} - 2^2$	$r=3$
$+6^2 - \cancel{3^2}$	$r=4$
$+7^2 - \cancel{4^2}$	$r=5$
$\vdots$	
$+\cancel{(n-2)^2} - (n-5)^2$	$r=n-4$
$+\cancel{(n-1)^2} - (n-4)^2$	$r=n-3$
$+(n)^2 - \cancel{(n-3)^2}$	$r=n-2$
$+(n+1)^2 - \cancel{(n-2)^2}$	$r=n-1$
$+(n+2)^2 - \cancel{(n-1)^2}]$	$r=n$

$$= (n^2 + (n+1)^2 + (n+2)^2 - 1 - 4)\,\frac{1}{3}$$

$$= (n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 - 1 - 4)\,\frac{1}{3}$$

$$= (3n^2 + 6n)\,\frac{1}{3}$$

$$= \frac{1}{3}(3n(n+2))$$

$$= n(n+2)$$

10 A curve, C_1, has equation

$$y = \frac{3x+5}{2x-7}$$

a) Write down the equations of the asymptotes of the curve C_1.

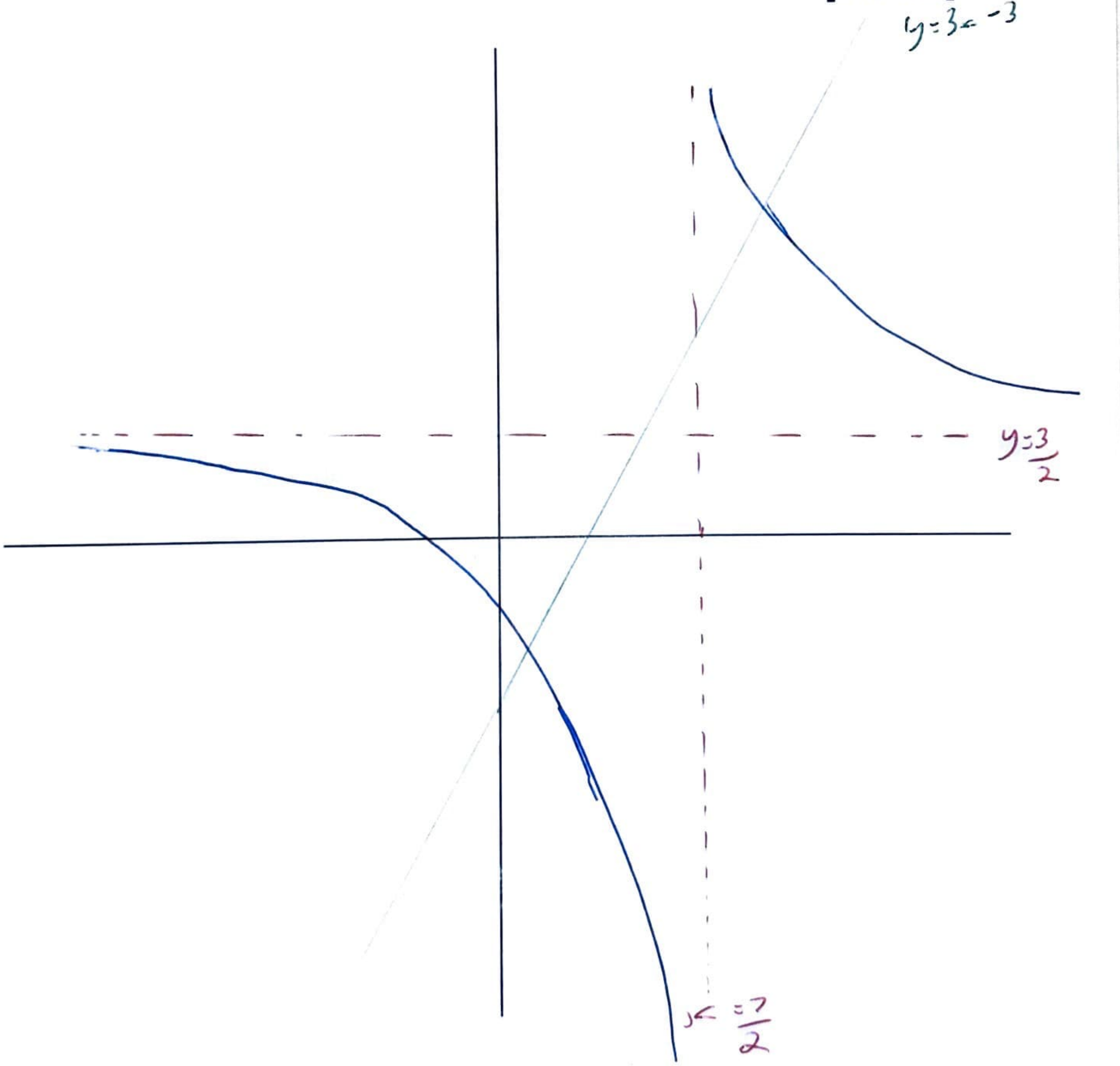
[2 marks]

$x = \frac{7}{2}$

$y = \frac{3}{2}$

b) Sketch, on the axes below the graph of C_1, indicating any intersections with the axes.

[3 marks]



c) Hence, solve the inequality

$$\frac{3x+5}{2x-7} > 3x - 3$$

[4 marks]

With reference to the graph, solving

$$\frac{3x+5}{2x-7} = 3x-3$$

$$3x+5 = (3x-3)(2x-7)$$

$$3x+5 = 6x^2 - \cancel{3x}\ 6x - 21x + 21$$

$$\Rightarrow 6x^2 - 30x + 16 \quad (*) \quad \cancel{\text{solving}}$$

Solving $(*)$

$$x = \frac{15+\sqrt{129}}{6} \quad \text{or} \quad x = \frac{15-\sqrt{129}}{6}$$

Hence,

$$\frac{3x+5}{2x-7} > 3x-3$$

When,

$$x < \frac{15-\sqrt{129}}{6} \quad \text{or} \quad \frac{7}{2} < x < \frac{15-\sqrt{129}}{6}$$

11 The roots of the polynomial

$$p(x) = 27x^3 - 78x^2 + cx - 8$$

are in a geometric progression. Find the value of c.

[5 marks]

$27x^3 - 78x^2 + cx - 8$

Let $\alpha = \frac{a}{r}$, $\beta = a$, and $\gamma = ar$, then by Viète's formulae,

$$\sum \alpha = \frac{78}{27}$$

$$\sum \alpha\beta = \frac{c}{27}$$

$$\sum \alpha\beta\gamma = \frac{8}{27}$$

Hence,

$$\sum \alpha = \frac{a}{r} + a + ar$$

$$\Rightarrow \frac{a}{r} + a + ar = \frac{78}{27}$$

$$\Rightarrow \frac{a}{r}(1 + r + r^2) = \frac{78}{27} \quad (1)$$

$$\sum \alpha\beta = \frac{a}{r} \times a + \frac{a}{r} \times ar + a \times ar$$

So $\Rightarrow \frac{a^2}{r} + a^2 + a^2 r = \frac{c}{27} \quad (2)$

and

$$\sum \alpha\beta\gamma = \frac{a}{r} \times a \times ar = a^3 \quad (3)$$

Hence $a^3 = \frac{8}{27}$

$$\Rightarrow a = \frac{2}{3}$$

Using (1)

$$\frac{2}{3r}(1+r+r^2) = \frac{78}{27}$$

$$\Rightarrow 54(1+r+r^2) = 234r$$

$$\Rightarrow 54r^2 - 180r + 54 = 0$$

$$\Rightarrow r = 3 \text{ or } \frac{1}{3}$$

Using $r = 3$ (wlog) and $a = \frac{2}{3}$ in (2)

$$\frac{\left(\frac{2}{3}\right)^2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 \times 3^1 = \frac{c}{27}$$

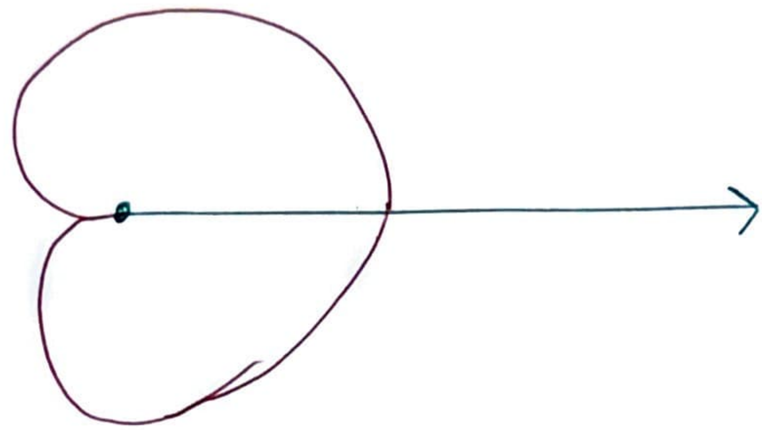
$$\Rightarrow \frac{52}{27} = \frac{c}{27}$$

So

$$c = 27$$

12 **a)** **(i)** Sketch the polar curve $r = 2 + 3\cos(\theta)$, $0 \leq \theta \leq 2\pi$., $r > 0$

[2 marks]



no inner loop as $r > 0$

(ii) What is the maximum value of r?

[1 mark]

maximum when $\cos(\theta) = 1$, so

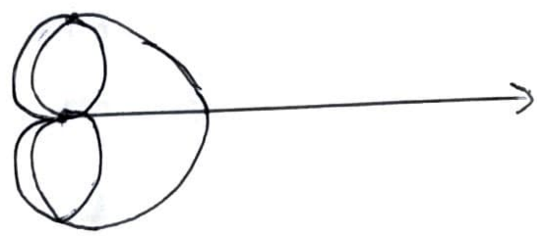
$r_{max} = 2 + 3$

$= 5$

b) The graph of the curve $r = 2\sin^2(\theta)$ is now added to the plot.

Find the polar coordinates of the points of intersection.

[4 marks]



At intersection points

$$2\sin^2\theta = 2 + 3\cos\theta$$

$$\Rightarrow 2(1-\cos^2\theta) = 2 + 3\cos(\theta)$$

$$\Rightarrow \quad 0 = 2\cos^2\theta + 3\cos\theta$$

$$= \cos(\theta)(2\cos(\theta)+3)$$

So $\cos(\theta) = 0$ or $\cos(\theta) = -\frac{3}{2}$.

Hence, $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

When $\theta = \frac{\pi}{2}$, $r = 2$

When $\theta = \frac{3\pi}{2}$, $r = -2$

Hence, points of intersection are $(2, \frac{\pi}{2})$ or $(-2, \frac{3\pi}{2})$

13 An indoor activity centre has two glider tracks (a bit like a zipline)

The floorpan of the centre is a rectangle of length 100 m (the $x-$direction) and width of 50 m ($y-$direction)

One glider track connects a point 40 m above the position with coordinate $A(100{,}0)$ with a point 25 m above the position with coordinate $B(0{,}40)$.

The other glider track begins at the point $C(100{,}50{,}35)$ with the point $D(10{,}20{,}20)$.

Stating any assumptions that you make, determine if it would be possible for two people who are 1.78m tall to collide when using these tracks.

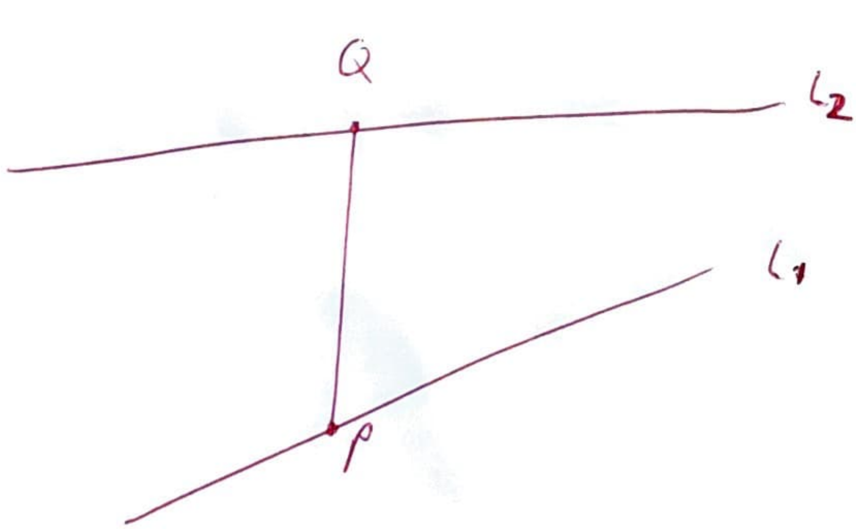
[8 marks]

Track connecting A and B:

$$L_1: \ \underset{\sim}{r_1} = \begin{pmatrix} 100 \\ 0 \\ 40 \end{pmatrix} + \lambda \begin{pmatrix} -100 \\ 40 \\ -15 \end{pmatrix}$$

Track connection C to D:

$$L_2: \ \underset{\sim}{r_2} = \begin{pmatrix} 100 \\ 50 \\ 35 \end{pmatrix} + \mu \begin{pmatrix} -90 \\ -30 \\ -15 \end{pmatrix}$$



So, vector $\overrightarrow{PQ}$:

$$\overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ}$$

$$= -\begin{pmatrix} 100-100\lambda \\ 40\lambda \\ 40-15\lambda \end{pmatrix} + \begin{pmatrix} 100-90\mu \\ 50-30\mu \\ 35-15\mu \end{pmatrix}$$

$$= \begin{pmatrix} 100\lambda - 90\mu \\ 50-40\lambda-30\mu \\ -5+15\lambda-15\mu \end{pmatrix}$$

PQ is $\perp$ to L_1, so

$$\begin{pmatrix} 100\lambda - 90\mu \\ 50-40\lambda-30\mu \\ -5+15\lambda-15\mu \end{pmatrix} \cdot \begin{pmatrix} -100 \\ 40 \\ -15 \end{pmatrix} = 0$$

$$\Rightarrow -10000\lambda + 9000\mu + 2000 - 1600\lambda - 1200\mu + 75 - 225\lambda + 225\mu = 0$$

$$\Rightarrow \cancel{-12800\lambda} \quad -11825\lambda + 8025\mu = -2075 \quad (1)$$

PQ is also $\perp$ to L_2, so

$$\begin{pmatrix} 100\lambda - 90\mu \\ 50-40\lambda-30\mu \\ -5+15\lambda-15\mu \end{pmatrix} \cdot \begin{pmatrix} -90 \\ -30 \\ -15 \end{pmatrix} = 0$$

$$-9000\lambda + 8100\mu - 1500 + 1200\lambda + 900\mu + 75 - 225\lambda + 225\mu = 0$$

$$\Rightarrow -8025\lambda + 9225\mu = 1425 \quad (2)$$

Solving ① and ②

$\lambda = \frac{453}{662}$, $\mu = 0.7497482377$

Hence,

$$\vec{PQ} = \begin{pmatrix} 0.951661628 1 \\ 0.1359516605 \\ -5.981873112 \end{pmatrix}$$

So

$$|\vec{PQ}| = \sqrt{0.9517^2 + 0.13595^2 + (-5.9819)^2}$$

$= 6.058625969$

$= 6.06$ to 3 sig fig.

So, assuming they both hang below their respective tracks they won't collide

14 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 3 & k \\ 0 & 1 \end{pmatrix}$$

a) Given that the image of $P(3, -2)$ is $P'(5, -2)$ find the value of k.

[2 marks]

$$\begin{pmatrix} 3 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$\Rightarrow 3 \times 3 - 2 \times k = 5$

$\Rightarrow 9 - 2k = 5$

$2k = 4$

$k = 2$

For the remainder of the question use the value of k found in (a).

b) For the matrix $\mathbf{A}$ given above verify that $\left(\mathbf{A}^{-1}\right)^{T} = \left(\mathbf{A}^{T}\right)^{-1}$

[4 marks]

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

$|A| = 3 \times 1 - 2 \times 0 = 3$

So. $A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$

So $(A^{-1})^{T} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$

and

$$A^T = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$$

so

$$(A^T)^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$$

Hence, we have verified that for this value of A,

$$(A^T)^{-1} = (A^{-1})^T$$

c) A shape of area 4 cm2 is transformed by the matrix $\mathbf{A}^3$. What is the area of the resulting shape?

[1 mark]

$3^3 \times 4 = 108 \text{cm}^2$

d) By calculating $\mathbf{A}^2$ and $\mathbf{A}^3$ make a conjecture about the form of $\mathbf{A}^n$ and prove it.

[5 marks]

$$A^2 = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 9 & 8 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 9 & 8 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 27 & 26 \\ 0 & 1 \end{pmatrix}$$

Conjecture: $\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 3^n-1 \\ 0 & 1 \end{pmatrix}$

Step 1: Let $P(n)$ be the statement

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 3^n-1 \\ 0 & 1 \end{pmatrix}$$

When $n = 1$

$$LHS = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

$$RHS = \begin{pmatrix} 3^1 & 3^1-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

So $P(1)$ is true.

Step 2: We assume $P(k)$ is true, i.e.

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 3^k-1 \\ 0 & 1 \end{pmatrix}$$

Step 3: For $n = k+1$, consider,

$$\begin{pmatrix}3 & 2\\ 0 & 1\end{pmatrix}^{k+1} = \begin{pmatrix}3 & 2\\ 0 & 1\end{pmatrix}^{k}\begin{pmatrix}3 & 2\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}3^k & 3^k-1\\ 0 & 1\end{pmatrix}\begin{pmatrix}3 & 2\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}3\times 3^k & 2\times 3^k + 3^k - 1\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}3^{k+1} & 3^{k+1}-1\\ 0 & 1\end{pmatrix}$$

Step 4: $P(1)$ is true, and if $P(k)$ is true then so is $P(k+1)$. Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$.

e) Let $\mathbf{B}$ be a general non-singular $n \times n$ matrix. Prove the general result,

$$\left(\mathbf{B}^{-1}\right)^T = \left(\mathbf{B}^T\right)^{-1}$$

[4 marks]

Let B^{-1} be the inverse of B, which exists since B is non-singular and square.

By definition,

$$B^{-1}B = I$$

$$\Rightarrow \quad (B^{-1}B)^T = I^T$$

So $\quad B^T(B^{-1})^T = I$

Similarly, if $BB^{-1} = I$, then

$$(BB^{-1})^T = I^T$$

$$\Rightarrow \quad (B^{-1})^T B^T = I$$

Hence,

$\cancel{(B)}$ $(B^{-1})^T$ is the ~~tr~~ inverse of B^T

and so

$$(B^T)^{-1} = (B^{-1})^T$$