Dr Bennison Maths

MATHEMATICS EDUCATION EXPERT

AQA A-Level Further Maths 2026 Paper 2

Do not turn over the page until instructed to do so.

This assessment is out of 100 marks and you will be given 120 minutes.

When you are asked to by your teacher write your **full name** below

Name:

Total Marks: **/ 100**

1 The matrix $\mathbf{M}$ represents a rotation about the y-axis.

$$\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ a & 0 & b \end{pmatrix}$$

Which of the following pairs is correct?

a) $a = -\dfrac{1}{2}.\ b = -\dfrac{\sqrt{3}}{2}$

b) $a = -\dfrac{1}{2}.\ b = \dfrac{\sqrt{3}}{2}$ ✓

c) $a = \dfrac{1}{2}.\ b = -\dfrac{\sqrt{3}}{2}$

d) $a = \dfrac{1}{2}.\ b = \dfrac{\sqrt{3}}{2}$

[1 mark]

2 Given that $y = \operatorname{cosech}(x)$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

a) $\operatorname{sech}(x)\tanh(x)$

b) $-\operatorname{sech}(x)\tanh(x)$

c) $\operatorname{cosech}(x)\coth(x)$

d) $-\operatorname{cosech}(x)\coth(x)$ ✓

[1 mark]

3 What is the geometric configuration for the system of equations below?

$$\begin{cases} x + 3y - 7z & = 8 \\ 2x + 0y - 2z & = -2 \\ x - y + z & = -4 \end{cases}$$

a) Unique solution (planes intersections at a single point)

b) Sheaf

c) Triangular prism

d) The two planes are parallel

[1 mark]

$$\underbrace{\begin{pmatrix} 1 & 3 & -7 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ -4 \end{pmatrix}$$

$\det(A) = 1 \times -2 - 3 \times 4 - 7 \times -2 = 0$ so not unique

No two planes are parallel

$x + 3y - 7z = 8$ (1)

$2x - 2z = -2$ (2)

$x - y + z = -4$ (3)

(2) $\Rightarrow x = z - 1$

(1) (3) $\Rightarrow 4y - 8z = 12 \Rightarrow y = 3 + 2z$

Then $z - 1 + 3(3 + 2z) - 7z = 8$. hence consistent, so sheaf

4 Show, fully justifying your answer, that

$$\int_{\infty}^{\infty} x\mathrm{e}^{-x^2}\,\mathrm{d}x = 0$$

[5 marks]

Consider $I_1 = \int_{-\infty}^{0} x e^{-x^2} dx$

$= \lim_{t \to -\infty} \int_{t}^{0} x e^{-x^2} dx$

$= \lim_{t \to \infty} \left[\frac{-1}{2} e^{-x^2} \right]_t^0$

$= \lim_{t \to \infty} \left[\frac{-1}{2} + \frac{1}{2} e^{-t^2} \right]$

$= \frac{-1}{2}$

Consider $I_2 = \int_{0}^{\infty} x e^{-x^2} dx$

$= \lim_{t \to \infty} \int_{0}^{t} x e^{-x^2} dx$

$= \lim_{t \to \infty} \left[-\frac{1}{2} e^{-x^2} \right]_0^t$

$= \lim_{t \to \infty} \left(-\frac{1}{2} e^{-t^2} + \frac{1}{2} \right)$

$= \frac{1}{2}$

Hence,

$I = \int_{-\infty}^{\infty} x e^{-x^2} dx$

$= \frac{-1}{2} + \frac{1}{2}$

$= 0$

5 Let $\mathbf{M}$ be the matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ -\frac{1}{2} & 3 \end{pmatrix}$

a) Find the eigenvalue-eigenvector pairs for the matrix $\mathbf{M}$

[4 marks]

$$\det(M - \lambda I) = |M - \lambda I|$$

$$= \begin{vmatrix} 2-\lambda & 0 \\ -\frac{1}{2} & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(3-\lambda)$$

So

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 3$$

When $\lambda_1 = 2$:

Consider

$$\begin{pmatrix} 2 & 0 \\ -\frac{1}{2} & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\Rightarrow \quad 2x = 2x$$

$$-\frac{1}{2}x + 3y = 2y$$

$$\Rightarrow \quad y = \frac{1}{2}x, \quad x \text{ can be anything}$$

Let

$$\underset{\sim}{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

When $\lambda_2 = 3$: Consider $\begin{pmatrix} 2 & 0 \\ -\frac{1}{2} & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \quad 2x = 3x$

$-\frac{1}{2}x + 3y = 3y$

$\Rightarrow \quad x = 0$

$y = 1$

Let $\underset{\sim}{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So the eigenvalue-eigenvector pairs are

$\lambda_1 = 2, \quad \underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 3, \quad \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) Without calculating $\mathbf{M}^{-1}$ state the eigenvalues and eigenvectors of $\mathbf{M}^{-1}$.

[2 marks]

$\lambda_1 = \frac{1}{2}, \quad \underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = \frac{1}{3}, \quad \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

6 The point $A(3,2,1)$ lies on a line which is perpendicular to the vectors $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

a) Find the Cartesian equation of the line.

[5 marks]

Direction vector of the line is perpendicular to $\underline{u}$ and $\underline{v}$, so

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix} = \underline{i}\begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$$

$$= 4\underline{i} - 5\underline{j} - 7\underline{k}$$

So, vector equation of the line is

$$\underline{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$$

Hence,

$$x = 3 + 4\lambda \Rightarrow \lambda = \frac{x-3}{4}$$

$$y = 2 - 5\lambda \Rightarrow \lambda = \frac{y-2}{-5} = \frac{2-y}{5}$$

$$z = 1 - 7\lambda \Rightarrow \lambda = \frac{z-1}{-7} = \frac{1-z}{7}$$

Hence Cartesian equation is

$$\frac{x-3}{4} = \frac{2-y}{5} = \frac{1-z}{7}$$

b) Show that the point $P(15, -13, -20)$ lies on the line and find the distance $|AP|$.

[3 marks]

Consider

$$\begin{pmatrix} 3 + 4\lambda \\ 2 - 5\lambda \\ 1 - 7\lambda \end{pmatrix} = \begin{pmatrix} 15 \\ -13 \\ -20 \end{pmatrix}$$

$\Rightarrow$ $\lambda = 3$
$\lambda = 3$
$\lambda = 3$

Since $\lambda = 3$ for the x, y, z components, P lies on the line.

$$|AP| = \sqrt{(12)^2 + (-15)^2 + (-14)^2}$$
$$= \sqrt{761}$$

7 Let $I_n = \int \tanh^n(x)\ \mathrm{d}x$.

a) Prove, that for $n \geq 2$,

$$I_n = I_{n-2} - \frac{1}{n-1}\tanh^{n-1}(x)$$

[4 marks]

$$I_n = \int \tanh^n(x)\,dx$$

$$= \int \tanh^{n-2}(x)\tanh^2(x)\,dx$$

$$= \int \tanh^{n-2}(x)\left(1 - \operatorname{sech}^2(x)\right)dx$$

$$= \int \tanh^{n-2}(x)\,dx - \int \tanh^{n-2}(x)\operatorname{sech}^2(x)$$

$$= I_{n-2} - \frac{1}{n-1}\tanh^{n-1}(x)$$

Hence, $I_n = I_{n-2} - \frac{1}{n-1}\tanh^{n-1}(x)$

b) Hence, find I_6

[3 marks]

~~H~~ Using (a)

$$I_6 = I_4 - \frac{1}{5}\tanh^5(x)$$

$$= I_2 - \frac{1}{3}\tanh^3(x) - \frac{1}{5}\tanh^5(x)$$

$$= I_0 - \tanh(x) - \frac{1}{3}\tanh^3(x) - \frac{1}{5}\tanh^5(x)$$

but $I_0 = \int \tanh^0(x)\,dx$

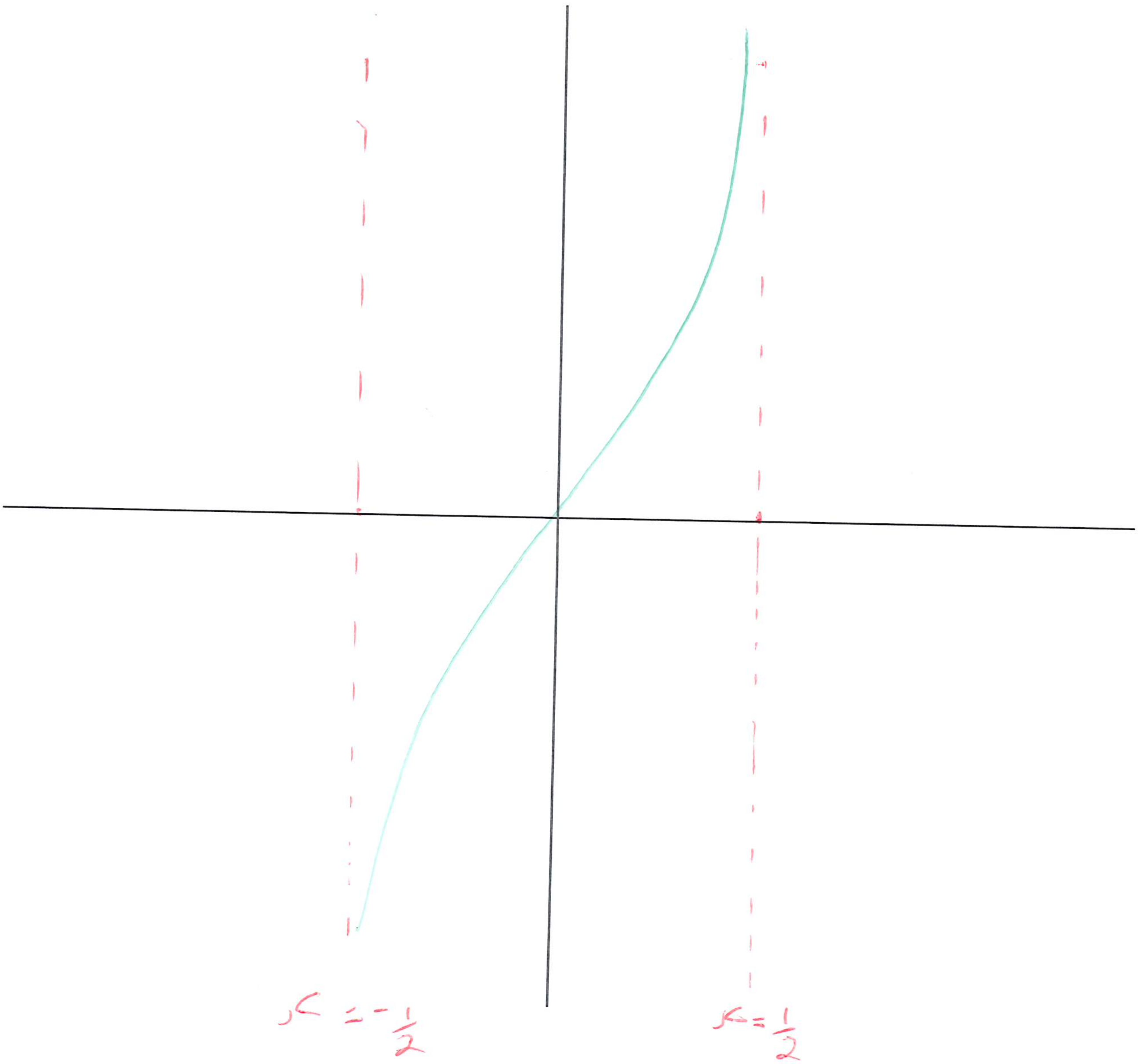
$$= \int 1\,dx$$

$$= x$$

So, $I_6 = x - \tanh(x) - \frac{1}{3}\tanh^3(x) - \frac{1}{5}\tanh^5(x)$

8

a) Sketch on the axes below $y = \mathrm{artanh}(2x)$ and state its domain and range.



[4 marks]

b) Derive the logarithmic form of $y = \text{artanh}(2x)$

[4 marks]

Let $y = \text{artanh}(2x)$

$\Rightarrow \quad \tanh(y) = 2x$

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = 2x$$

$$e^y - e^{-y} = 2x(e^y + e^{-y})$$

$$e^y - e^{-y} = 2xe^y + 2xe^{-y}$$

$$e^y(1 - 2x) = e^{-y}(1 + 2x)$$

$$\frac{e^y}{e^{-y}} = \frac{1+2x}{1-2x}$$

$$e^{2y} = \frac{1+2x}{1-2x}$$

$$2y = \ln\left|\frac{1+2x}{1-2x}\right|$$

$$y = \frac{1}{2}\ln\left|\frac{1+2x}{1-2x}\right|$$

c) Solve, giving your answer in an exact form,

$$6 + \tanh(x) = 12\operatorname{sech}^2(x)$$

[4 marks]

Using $\operatorname{sech}^2(x) = 1 - \tanh^2(x)$,

$$6 + \tanh(x) = 12\operatorname{sech}^2(x)$$

$$\Rightarrow \quad 6 + \tanh(x) = 12(1 - \tanh^2(x))$$

$$6 + \tanh(x) = 12 - 12\tanh^2(x)$$

$$\Rightarrow \quad 12\tanh^2(x) + \tanh(x) - 6 = 0$$

$$(3\tanh(x) - 2)(4\tanh(x) + 3) = 0$$

So

$$\tanh(x) = \frac{2}{3} \quad \text{or} \quad \tanh(x) = -\frac{3}{4}$$

Hence,

$$x = \frac{1}{2}\ln\left|\frac{1 + 2/3}{1 - 2/3}\right|$$

$$= \frac{1}{2}\ln|5|$$

and

$$x = \frac{1}{2}\ln\left|\frac{1 - 3/4}{1 + 3/4}\right|$$

$$= \frac{1}{2}\ln\left|\frac{1}{7}\right|$$

$$= -\frac{1}{2}\ln|7|$$

9

The function f is defined by

$$f(x) = \frac{x^2 - 4x + 5}{x - 3}, \qquad x \in \mathbb{R}, x \neq 3$$

a) Find the interval (a, b) in which $f(x)$ does not take a value.

Fully justify your answer.

[5 marks]

Assume $y = k$ meets $f(x)$ then

$$k = \frac{x^2 - 4x + 5}{x - 3}$$

$$\Rightarrow k(x-3) = x^2 - 4x + 5$$

$$\Rightarrow kx - 3k = x^2 - 4x + 5$$

$$\Rightarrow x^2 + (-4-k)x + (5+3k) = 0$$

We are looking for when the discriminant is < 0.

$\Delta < 0 \Rightarrow$

$$(-4-k)^2 - 4 \times 1 \times (5 + 3k) < 0$$

$$\Rightarrow k^2 - 4k - 4 < 0$$

So critical points are $k = 2 + 2\sqrt{2}$ and $2 - 2\sqrt{2}$.

Hence,

$$2 - 2\sqrt{2} < k < 2 + 2\sqrt{2}$$

So, $f(x)$ doesn't take a value in the interval $(2-2\sqrt{2}; 2+2\sqrt{2})$

b) Find the coordinates of the two stationary points of the graph $y = f(x)$.

[3 marks]

When $k = 2 - 2\sqrt{2}$,

$$x^2 + (-6 + 2\sqrt{2}) + (5 + 3(2 - 2\sqrt{2})) = 0$$

$$\Rightarrow \quad x = 3 - \sqrt{2}$$

So stationary point at $(3 - \sqrt{2}, 2 - 2\sqrt{2})$

When $k = 2 + 2\sqrt{2}$

$$x^2 + (-6 - 2\sqrt{2}) + (5 + 3(2 + \sqrt{2})) = 0$$

$$\Rightarrow x = 3 + \sqrt{2}$$

So stationary point at $(3 + \sqrt{2}, 2 + \sqrt{2})$

c) Show that the curve has an oblique asymptote and find it's equation.

[2 marks]

$$\begin{array}{r|l} & x - 1 \\ \hline x - 3 & x^2 - 4x + 5 \\ & \underline{x^2 - 3x} \\ & -x + 5 \\ & \underline{-x + 3} \\ & 2 \end{array}$$

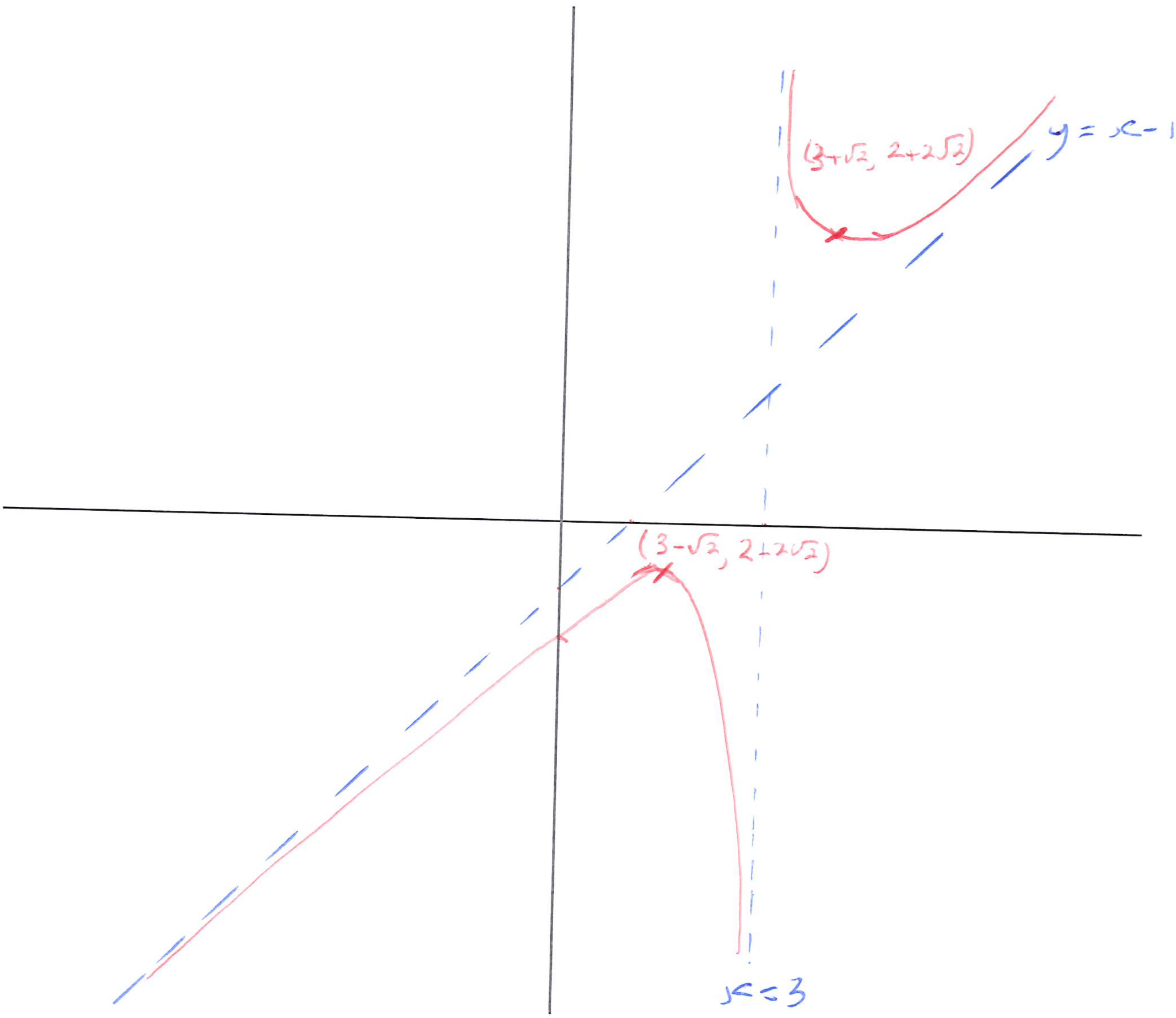
Hence oblique asymptote with equation $y = x - 1$, and $f(x) = x - 1 + \frac{2}{x-3}$

d) Sketch the graph of $y = f(x)$ on the axes below.

[4 marks]

Vertical asymptote at $x = 3$

When $x = 0$, $y = \frac{-5}{3}$



10 **a)** Solve, given that at $x = 0, y = 2$, the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan(x) = 2\sin(x)$$

[6 marks]

Consider the integrating factor,

$$IF = e^{\int \tan(x)\,dx}$$

$$= e^{-\int \tan(x)\,dx}$$

$$= e^{-\ln|\sec(x)|}$$

$$= e^{\ln|\cos(x)|}$$

$$= \cos(x)$$

Hence,

$$\cos(x)\frac{dy}{dx} - y\cos(x)\tan(x) = 2\sin(x)\cos(x)$$

$$\frac{d}{dx}\left(y\cos(x)\right) = \sin(2x)$$

Hence

$$\int \frac{d}{dx}(y\cos(x))\,dx = \int \sin(2x)\,dx$$

$$\Rightarrow \quad y\cos(x) = -\frac{1}{2}\cos(2x) + C$$

So,

$$y = -\frac{\cos(2x)}{2\cos(x)} + C$$

when $x = 0$, $y = 2$, so

$$2 = \frac{-\cos(0)}{2\cos(0)} + C$$

$$\Rightarrow C = 2 + \frac{1}{2} = \frac{5}{2}$$

Hence,

$$y = \frac{-\cos(2x)}{2\cos(x)} + \frac{5}{2}.$$

b) George uses Euler's method with a stepwise of $h = 0.1$ to find an estimate of the value of y when $x = 0.3$ for the differential equation defined in (a).

Assuming he performs the calculations correctly, if he begins with $x_0 = 1.1$ and $y_0 = 2.0075$, what does George calculate $y(0.3)$ to be.

[7 marks]

$$\frac{dy}{dx} - y\tan(x) = 2\sin(x)$$

$$\Rightarrow \frac{dy}{dx} = \underbrace{y\tan(x) + 2\sin(x)}_{=f(x)}$$

Euler's method: $x_{r+1} = x_r + h$

$$y_{r+1} = y_r + hf(x_r, y_r)$$

r	x	y
0	1.1	2.0075
1	1.2	2.580166973
2	1.3	3.430232857

11 Consider the following inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = e^{-3x} + \cos(x) \qquad (\dagger)$$

a) Find the complementary function for this differential. equation.

[3 marks]

Auxiliary equation:

$$m^2 + 2m - 3 = 0$$
$$(m-1)(m+3) = 0$$

So $m = 1,\ m = -3$

The complementary function is

$$y = Ae^{x} + Be^{-3x}$$

b) Lucy decides to try $y = \lambda e^{-3x} + \mu \cos(x)$ as a particular integral.

Give two reasons why this is not a suitable choice of trial function.

[2 marks]

- e^{-3x} appears in the complementary function so we need to try xe^{-3x}
- If $\cos(x)$ appears in the right hand side we need to try $\mu\cos(x) + \gamma\sin(x)$ due to the cyclic derivative property of $\sin(x)$ and $\cos(x)$

c) Given that at $x = 0, y = \dfrac{14}{5}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{17}{20}$ solve this differential equation.

[6 marks]

For PI try

$y = \alpha x e^{-3x} + \beta\cos(x) + \gamma\sin(x)$

$\frac{dy}{dx} = -3\alpha e^{-3x}x + \alpha e^{-3x} - \beta\sin(x) + \gamma\cos(x)$

$\frac{d^2y}{dx^2} = 9\alpha e^{-3x}x - 3\alpha e^{-3x} - 3\alpha e^{-3x} - \beta\cos(x) - \gamma\sin(x)$

Sub into (†)

$3\alpha e^{-3x}(3x-2) - \beta\cos(x) - \gamma\sin(x)$
$+2(-3\alpha x e^{-3x} + \alpha e^{-3x} - \beta\sin(x) + \gamma\cos(x))$
$-3(\alpha x e^{-3x} + \beta\cos(x) + \gamma\sin(x)) = e^{-3x} + \cos(x)$

$\underline{e^{-3x}|}$ $9x\alpha - 6\alpha + 2\alpha - 6x\alpha - 3\alpha x = 1$

$\Rightarrow -4\alpha = 1$

$\Rightarrow \alpha = -\frac{1}{4}$

$\underline{\cos(x)|}$ $-\beta + 2\gamma - 3\beta = 1$ $\Rightarrow$

$\Rightarrow 2\gamma - 4\beta = 1$ (1)

$\underline{\sin(x)|}$ $-\gamma - 2\beta - 3\gamma = 0$

$\Rightarrow -4\gamma - 2\beta = 0$ (2)

Solving (1) and (2) $\gamma = \frac{1}{10}, \beta = -\frac{1}{5}$

Hence, general solution is,

$$y = Ae^{x} + Be^{-3x} - \frac{1}{4}xe^{-3x} - \frac{1}{5}\cos(x) + \frac{1}{10}\sin(x)$$

$$y' = Ae^{x} - 3Be^{-3x} - \frac{1}{4}xe^{-3x} + \frac{3}{4}xe^{-3x} + \frac{1}{5}\sin(x) + \frac{1}{10}\cos(x)$$

Since $y(0) = \frac{14}{5}$

$$A + B - \frac{1}{5} = \frac{14}{5}$$

$$\Rightarrow A + B = 3 \qquad (3)$$

and $y'(0) = \frac{17}{20}$

$$A - 3B - \frac{1}{4} + \frac{1}{10} = \frac{17}{20}$$

$$\Rightarrow A - 3B = 1 \qquad (4)$$

Solving (3) and (4)

$$A = \frac{5}{2}, \quad B = \frac{1}{2}$$

Hence, the general solution is,

$$y = \frac{5}{2}e^{x} + \frac{1}{2}e^{-3x} - \frac{1}{4}xe^{-3x} - \frac{1}{5}\cos(x) + \frac{1}{10}\sin(x)$$

12

a) Use De Moivre's theorem to show that

$$\tan(6\theta) = \frac{6\tan(\theta) - 20\tan^3(\theta) + 6\tan^5(\theta)}{1 - 15\tan^2(\theta) + 15\tan^4(\theta) - \tan^6(\theta)}$$

[6 marks]

$$(\cos(\theta) + i\sin(\theta))^6 = \cos(6\theta) + i\sin(6\theta)$$

By the Binomial expansion,

$$\cos(6\theta) + i\sin(6\theta) = \cos^6(\theta) + \binom{6}{1}\cos^5\theta \, i\sin(\theta)$$
$$+ \binom{6}{2}\cos^4(\theta)(i\sin(\theta))^2$$
$$+ \binom{6}{3}\cos^3(\theta)(i\sin(\theta))^3 + \binom{6}{4}\cos^2(\theta)(i\sin(\theta))^4$$
$$+ \binom{6}{5}\cos(\theta)(i\sin(\theta))^5 + (i\sin(\theta))^6$$
$$= \cos^6(\theta) + 6i\cos^5(\theta)\sin^5(\theta) - 15\cos^4(\theta)\sin^2(\theta)$$
$$-20i\sin^3(\theta)\cos^3(\theta) + 15\cos^2(\theta)\sin^4(\theta)$$
$$+6i\cos(\theta)\sin^5(\theta) - \sin^6(\theta)$$

Real

$$\cos(6\theta) = \cos(6\theta) - 15\cos^4(\theta)\sin^2(\theta) + 15\cos^2(\theta)\sin^4(\theta)$$
$$-\sin^6(\theta)$$

Img

$$\sin(6\theta) = 6\cos^5(\theta)\sin(\theta) - 20\sin^3(\theta)\cos^3(\theta) + 6\cos(\theta)\sin^5(\theta)$$

So,

$$\tan(6\theta) = \frac{\sin(6\theta)}{\cos(6\theta)}$$

$$= \frac{6\cos^5(\theta)\sin(\theta) - 20\sin^3(\theta)\cos^3(\theta) + 6\cos(\theta)\sin^5(\theta)}{\cos^6(\theta) - 15\cos^4(\theta)\sin^2(\theta) + 15\cos^2(\theta)\sin^4(\theta) - \sin^6(\theta)}$$

$$= \frac{\frac{6\cos^5(\theta)\sin(\theta)}{\cos^6(\theta)} - \frac{20\sin^3(\theta)\cos^3(\theta)}{\cos^6(\theta)} + \frac{6\cos(\theta)\sin^5(\theta)}{\cos^6(\theta)}}{\frac{\cos^6(\theta)}{\cos^6(\theta)} - \frac{15\cos^4(\theta)\sin^2(\theta)}{\cos^6(\theta)} + \frac{15\cos^2(\theta)\sin^4(\theta)}{\cos^6(\theta)} - \frac{\sin^6(\theta)}{\cos^6(\theta)}}$$

$$= \frac{6\tan(\theta) - 20\tan^3(\theta) + 6\tan^5(\theta)}{1 - 15\tan^2(\theta) + 15\tan^4(\theta) - \tan^6(\theta)}$$

b) Hence, show that $t = \tan\left(\frac{\pi}{24}\right)$ is a solution of the polynomial equation $t^4 + 8t^3 + 2t^2 - 8t + 1 = 0$

[4 marks]

Let $t = \tan(\theta)$, if $\tan(6\theta) = 1$

$$1 = \frac{6t - 20t^3 + 6t^5}{1 - 15t^2 + 15t^4 - t^6}$$

$$\Rightarrow 1 - 15t^2 + 15t^4 - t^6 = 6t - 20t^3 + 6t^5$$

$$\Rightarrow t^6 + 6t^5 - 15t^4 - 20t^3 + 15t^2 + 6t - 1 = 0 \quad (*)$$

So, $\tan(6\theta) = 1 \Rightarrow 6\theta = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{24}$

Factorising $(*)$,

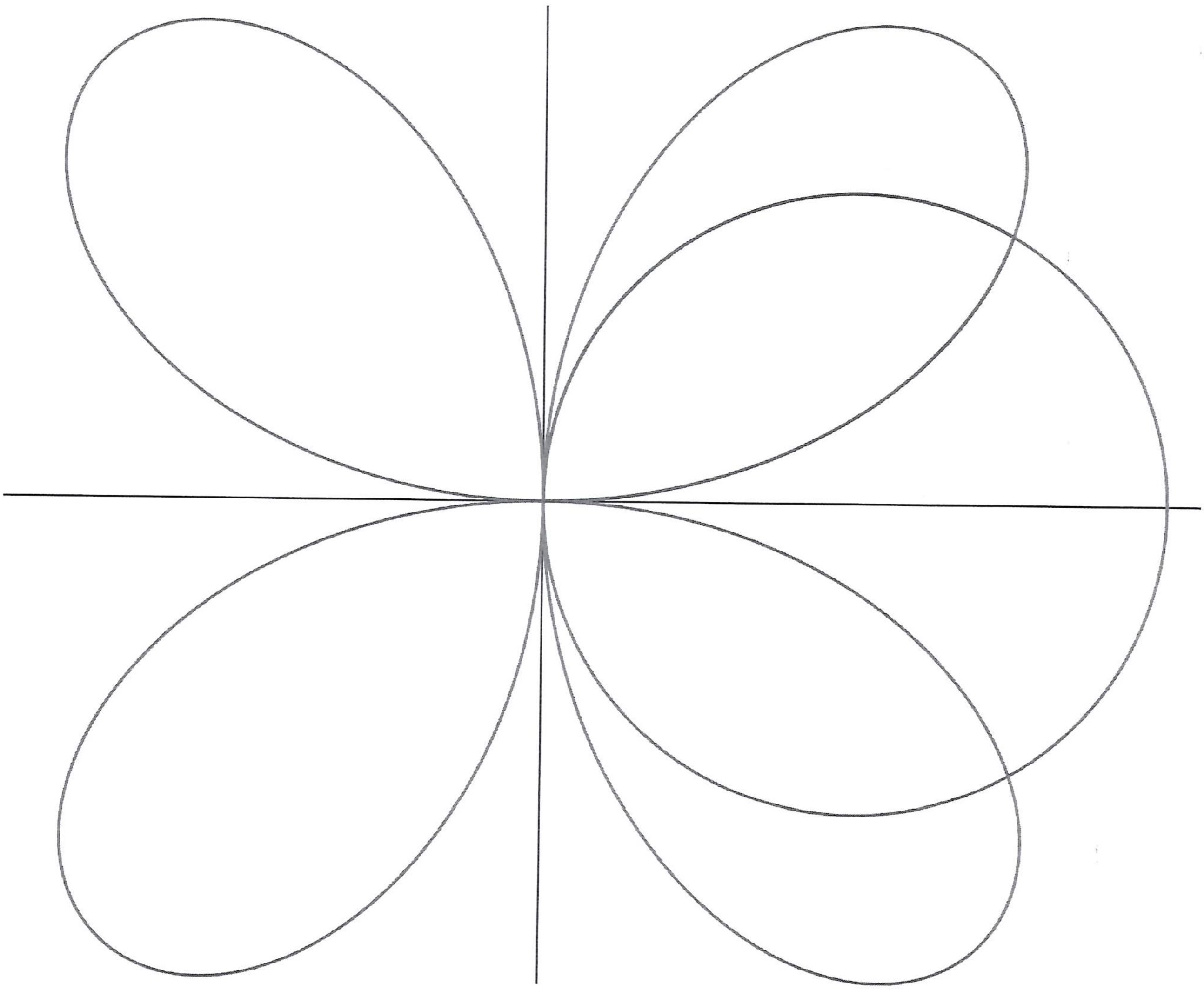
$$t^6 + 6t^5 - 15t^4 - 20t^3 + 15t^2 + 6t - 1 = (t^2 - 2t - 1)(t^4 + 8t^3 + 2t^2 - 8t + 1)$$

but $\tan\left(\frac{\pi}{24}\right)$ is not a solution of $t^2 - 2t - 1 = 0$

So $t = \tan\left(\frac{\pi}{24}\right)$ is a solution of $t^4 + 8t^3 + 2t^2 - 8t + 1 = 0$

13

The diagram below shows the polar curve C_1 with equation
$r = 3\cos(\theta),\ 0 \leq \theta \leq \pi.$



It also shows part of the polar curve C_2 with equation $r = 3\sin(2\theta)$,
$0 \leq \theta \leq 2\pi.$

a) On the diagram above, complete the sketch of C_2

[2 marks]

See above

b) Show that the shaded area is $\frac{9\pi}{8} - \frac{27\sqrt{3}}{32}$.

[9 marks]

$$3\cos(\theta) = 3\sin(2\theta)$$
$$\cos(\theta) = 2\sin(\theta)\cos(\theta)$$
$$\Rightarrow \quad 0 = 2\sin(\theta)\cos(\theta) - \cos(\theta)$$
$$= \cos(\theta)(2\sin(\theta) - 1)$$

So $\cos(\theta) = 0 \Rightarrow \theta = \frac{\pi}{2}$

or $2\sin(\theta) - 1 = 0 \Rightarrow \sin(\theta) = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$$\text{Area} = \frac{1}{2}\int_{\pi/6}^{\pi/2} (3\cos(\theta))^2 \, d\theta$$
$$= \frac{9}{2}\int_{\pi/6}^{\pi/2} \cos^2(\theta)\, d\theta$$
$$= \frac{9}{2}\int_{\pi/6}^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta$$
$$= \frac{9}{2}\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_{\pi/6}^{\pi/2}$$
$$= \frac{9}{2}\left[\left(\frac{\pi}{4} + 0\right) - \left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)\right.$$

$$= \frac{9}{2}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$$

$$= \frac{9\pi}{12} - \frac{9\sqrt{3}}{16}$$

$$= \frac{3\pi}{4} - \frac{9\sqrt{3}}{16}$$

iii $\text{Area} = \frac{1}{2}\int_0^{\pi/6} (3\sin(2\theta))^2 \, d\theta$

$$= \frac{9}{2}\int_0^{\pi/6} \sin^2(2\theta) \, d\theta$$

$$= \frac{9}{2}\int_0^{\pi/6} \frac{1-\cos(4\theta)}{2} \, d\theta$$

$$= \frac{9}{2}\left[\frac{\theta}{2} - \frac{\sin(4\theta)}{8}\right]_0^{\pi/6}$$

$$= \frac{9}{2}\left[\left(\frac{\pi}{12} - \frac{\sqrt{3}}{16}\right) - (0-0)\right]$$

$$= \frac{9\pi}{24} - \frac{9\sqrt{3}}{32}$$

$$= \frac{3\pi}{8} - \frac{9\sqrt{3}}{32}$$

So, total area $= \frac{3\pi}{4} - \frac{9\sqrt{3}}{16} + \frac{3\pi}{8} - \frac{9\sqrt{3}}{32}$

$= \frac{9\pi}{8} - \frac{27\sqrt{3}}{32}$